Journal of Mathematical Analysis and Applications **233,** 102–115 (1999)
Article ID jmaa.1999.6269, available online at http://www.idealibrary.com on IDEAL®

Dimension of the Global Attractor for Strongly Damped Nonlinear Wave Equation

Shengfan Zhou*

Submitted by R. E. Showalter

Received February 10, 1998

The existence and estimate of the upper bound of the Hausdorff dimension of the global attractor for the strongly damped nonlinear wave equation with the Dirichlet boundary condition are considered by introducing a new norm in the phase space. The gained Hausdorff dimension decreases as the damping grows and remains small for large damping, which conforms to physical intuition. © 1999 Academic Press

Key Words: wave equation; global attractor; Hausdorff dimension; equivalent norm.

1. INTRODUCTION

Consider the strongly damped nonlinear wave equation with the Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \alpha\Delta\dfrac{\partial u}{\partial t} - \Delta u = f(u, u_t) + g, \; x \in \Omega, \; t > 0, \\ u(x,t)|_{x\in\partial\Omega} = 0, \; t > 0 \end{cases} \tag{1}$$

and the initial value conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \qquad x \in \Omega, \tag{2}$$

where $u = u(x,t)$ is a real-valued function on $\Omega \times [0, +\infty)$, Ω is an open bounded set of R^n with a smooth boundary $\partial\Omega$, $\alpha > 0$, $D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$, $f(u, v) \in C^1(R \times R; R)$, $g \in L^2(\Omega)$.

* Supported by the National Natural Science Foundation of China.

0022-247X/99 $30.00

For the system (1)–(2) where the function $f(u, u_t) = f(u) \in C^1(R; R)$ with $f'(u) \leq c_0$ for some constant $c_0 \geq 0$, $f(0) = 0$, and $\overline{\lim}_{|u| \to \infty}(f(u)/u) \leq 0$, Webb [3] considered its asymptotic behavior and proved that it is a gradient system. For the general abstract function $f = f(u, u_t)$, Massatt [4], and Hale [5] showed if the continuous semigroup of mapping $S(t)$: $\{u_0, u_1\} \mapsto \{u, u_t\}$ for $t \geq 0$, from $E = H_0^1(\Omega) \times L^2(\Omega)$ into itself, defined by system (1)–(2) (if it exists) is point dissipative and is a bounded map, then there is a global attractor for (1)–(2) in E.

In this paper, we consider the existence and Hausdorff dimensional estimate of the global attractor in E for system (1)–(2) where the function $f(u, v)$ satisfies

$$\begin{cases} f(u,v) \in C^1(R \times R; R), \qquad |f(u,v)| \leq k_0 + k_1 |u|^{\delta_0}, \\ \qquad \forall (u,v) \in R \times R, \\ |f_1(u,v)| \leq k_2, \qquad |f_2(u,v)| \leq k_3, \qquad \forall (u,v) \in R \times R, \end{cases} \tag{3}$$

and its partial derivatives $f_1(u, v)$, $f_2(u, v)$ satisfy:

$$\begin{cases} |f_1(u_1,v) - f_1(u_2,v)| \leq k_4 |u_1 - u_2|^{\delta_1} & \forall u_1, u_2, v \in R, \\ |f_1(u,v_1) - f_1(u,v_2)| \leq k_5 |v_1 - v_2|^{\delta_2}, & \forall u, v_1, v_2 \in R, \\ |f_2(u,v_1) - f_2(u,v_2)| \leq k_6 |v_1 - v_2|^{\delta_3}, & \forall u, v_1, v_2 \in R. \end{cases} \tag{4}$$

where $f_1(u, v) = (\partial f / \partial u)(u, v)$, $f_2(u, v) = (\partial f / \partial v)(u, v)$, $k_i > 0$, $i = 0, 1, \ldots, 6$, $0 < \delta_0 < 1$, $\delta_j > 0$, $j = 1, 2, 3$.

We obtain an upper bound of the Hausdorff dimension for the global attractor by introducing a new norm in the phase space E which is equivalent to the usual norm in E and by carefully estimating and splitting the positivity of the linear operator in the corresponding evolution equation of the first order in time. The gained upper bound of the Hausdorff dimension decreases as the damping α grows and remains small for large damping α, which conforms to physical intuition. The idea of using such a technique originates in Zhu and Zhou [6] and Wang and Zhu [7]. The main result is the following theorem.

THEOREM 1. (i) *If the function* $f(u, v)$ *satisfies conditions* (3), *then the semigroup defined by system* (1)–(2) *possess a global attractor in the space* $E = H_0^1(\Omega) \times L^2(\Omega)$.

(ii) *If the function* $f(u, v)$ *satisfies conditions* (3), (4), *and* $\alpha > (2k_3/\lambda_1)$, *then the Hausdorff dimension* $d_H(\mathfrak{B})$ *of the global attractor* $\mathfrak{B}$ *for* *system* (1)–(2) *in E satisfies*:

$$d_H(\mathfrak{B}) \leq \min\left\{ \ell | \ell \in N, \frac{1}{\ell}\sum_{j=1}^{\ell} \lambda_j^{-1} \leq \frac{2\delta\alpha\sigma}{k_2 + \varepsilon k_3} \right\}, \tag{5}$$

where

$$\sigma = \frac{\lambda_1 \alpha}{\alpha^2\lambda_1 + 4 + \alpha\sqrt{\alpha^2\lambda_1^2 + 4\lambda_1}}, \qquad \varepsilon = \frac{\alpha\lambda_1}{4 + 2\alpha^2\lambda_1},$$
$$\delta = \frac{\alpha\lambda_1 - 2k_3}{2\alpha(k_2 + \varepsilon k_3)} \tag{6}$$

and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$, $\lambda_m \to +\infty (m \to +\infty)$ *are the eigenvalues of operator* $-\Delta$ *with the Dirichlet boundary condition on* Ω.

(iii) *If the function* $f(u, u_t) = f(u) \in C^1(R, R)$ *is independent of* u_t *and satisfies condition* (3) *and* (4), *i.e.*,

$$|f(u)| \leq k_0 + k_1|u|^{\delta_0}, \qquad |f'(u)| \leq k_2,$$
$$|f'(u_1) - f'(u_2)| \leq k_4|u_1 - u_2|^{\delta_1}, \qquad \forall u, u_1, u_2 \in R, \tag{7}$$

in which $k_i > 0$, $i = 0, 1, 2, 4$, $0 < \delta_0 < 1$, $\delta_1 > 0$, *then the Hausdorff dimension* $d_H(\mathfrak{B})$ *of the global attractor* $\mathfrak{B}$ *for system* (1)–(2) *in E satisfies*

$$d_H(\mathfrak{B}) \leq \min\left\{ \ell | \ell \in N, \frac{1}{\ell}\sum_{j=1}^{\ell} \lambda_j^{-1} \leq \frac{\alpha\lambda_1\sigma}{k_2^2} \right\}. \tag{8}$$

It is easy to see from (5) and (8) that $d_H(\mathfrak{B})$ remains small for sufficiently large α because $(1/\ell)\sum_{j=1}^{\ell}\lambda_j^{-1}$ is a decreasing sequence with respect to ℓ and $(1/\ell)\sum_{j=1}^{\ell}\lambda_j^{-1} \to 0$ as $\ell \to +\infty$ but $(2\delta\alpha\sigma/(k_2 + \varepsilon k_3))$ or $(\alpha\lambda_1\sigma/k_2^2)$ is an increasing function of α for suitably large α and

$$\lim_{\alpha \to +\infty} \frac{2\delta\alpha\sigma}{k_2 + \varepsilon k_3} = \lim_{\alpha \to +\infty} \frac{\alpha\lambda_1\sigma}{k_2^2} = \frac{\lambda_1}{2k_2^2}.$$

So the Hausdorff dimension of the global attractor for (1)–(2) is uniformly bounded and is independent of the damping α if α is not very small.

2. PRELIMINARIES

It is known that the linear operator $A = -\Delta: D(A) \to L^2(\Omega)$ is self-adjoint positive and the eigenvalues $\{\lambda_i\}_{i \in N}$ of A satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \qquad \text{and } \lambda_m \to +\infty \quad \text{as } m \to +\infty.$$

Let

$$(u,v) = \int_\Omega uv\,dx, \qquad |u| = (u,u)^{1/2}, \qquad \forall u,v \in L^2(\Omega),$$

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v\,dx, \qquad \|u\| = ((u,u))^{1/2}, \qquad \forall u,v \in H_0^1(\Omega),$$

and

$$(y_1,y_2)_{H_0^1 \times L^2} = ((u_1,u_2)) + (v_1,v_2), \qquad |y|_{H_0^1\times L^2} = (y,y)_{H_0^1\times L^2}^{1/2}$$

$$\forall y_i = (u_i,v_i)^T, \qquad y = (u,v)^T \in H_0^1(\Omega) \times L^2(\Omega), \qquad i = 1,2,$$

denote the usual inner products and norms in $L^2(\Omega)$, $H_0^1(\Omega)$, and $H_0^1(\Omega) \times L^2(\Omega)$, respectively.

Let $u_t = v$, then (1)–(2) are equivalent to the following initial value problem in the space E:

$$\begin{cases} \dot{Y} = CY + F(Y), & x \in \Omega, \quad t > 0, \\ Y(0) = Y_0 = (u_0,u_1)^T \in E, \end{cases} \tag{9}$$

where

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & I \\ -A & -\alpha A \end{pmatrix}, \qquad F(Y) = \begin{pmatrix} 0 \\ f(u,u_t) + g \end{pmatrix} \tag{10}$$
$$D(C) = D(A) \times D(A).$$

Massatt in [4] proved that C in (10) is a sectorial operator on E and generates an analytic compact semigroup e^{Ct} on E for $t > 0$. By the assumptions (3), it is easy to see that the function $F(Y)$: $E \to E$ is continuously differentiable and globally Lipschitz continuous with respect to Y. By the classical semigroup theory concerning the existence and uniqueness of the solutions of differential equations [8], we have following lemma.

LEMMA 1. *Consider the initial value problem* (9) *on the Hilbert space E. If the function $f(u,v)$ satisfies conditions* (3), *then*

(i) *For any $Y_0 \in E$, there exists a unique function $Y(\cdot) = Y(\cdot, Y_0) \in C(R_+; E)$ such that $Y(0,Y_0) = Y_0$ and $Y(t)$ satisfies the integral equation*

$$Y(t,Y_0) = e^{Ct}Y_0 + \int_0^t e^{C(t-\tau)}F(Y(\tau))\,d\tau. \tag{11}$$

In this case, $Y(t)$ is called a mild solution of (9).

(ii) *If* $Y_0 \in D(C)$, *there exists* $Y(\cdot) \in C(R_+; D(C)) \cap C^1(R_+; E)$ *which satisfies* (9).

(iii) $Y(t, Y_0)$ *is jointly continuous in* t *and* Y_0.

For any $t \geq 0$, we introduce a map $S(t)$: $Y_0 \mapsto Y(t, Y_0)$, where $Y(t, Y_0)$ is the mild solution (or solution) of (9), then $\{S(t),\ t \geq 0\}$ is a continuous semigroup on E or on $D(C)$.

LEMMA 2. *Consider the linearized equation of* (1)–(2)

$$\begin{cases} U_{tt} - \alpha \Delta U_t - \Delta U = f_1(u, u_t)U + f_2(u, u_t)U_t, & \\ U(x,t)|_{x \in \partial\Omega} = 0, & t > 0, \\ U(x,0) = (\xi, \eta)^T, & x \in \Omega. \end{cases} \tag{12}$$

If the function $f(u, v)$ *satisfies conditions* (3) *and* (4), *then* (12) *is a well-posed problem in* E, *the mapping* $S(t)$ *defined in* (9) *is Fréchet differentiable on* E *for any* $t > 0$, *its differential at* $\varphi = (u_0, u_1)^T$ *is the linear operator on* E: $(\xi, \eta)^T \mapsto (U(t), V(t))^T$, *where* U *is the solution of* (12) *and* $V = U_t$.

Proof. It is clear from assumptions (3) that (12) is a well-posed problem in E.

We first consider the Lipschitz property of $S(t)$ on the bounded sets of E. Let $\varphi_0 = (u_0, u_1)^T \in D(C)$, $\tilde{\varphi}_0 = \varphi_0 + (\xi, \eta)^T = (u_0 + \xi, u_1 + \eta)^T \in D(C)$. It is known from Lemma 1 that the solutions $S(t)\varphi_0 = \varphi(t) = (u(t), u_t(t))^T \in D(C)$, $S(t)\tilde{\varphi}_0 = \tilde{\varphi}(t) = (\tilde{u}(t), \tilde{u}_t(t))^T \in D(C)$.

The difference $\psi = \tilde{u} - u$ satisfies

$$\psi_{tt} - \alpha \Delta \psi - \Delta \psi = f(\tilde{u}, \tilde{u}_t) - f(u, u_t). \tag{13}$$

Taking the scalar product of (13) with $\psi_t = \tilde{u}_t - u_t$ in $L^2(\Omega)$ and by the mean value theorem, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|\psi_t|^2 + \|\psi\|^2\right) + \alpha\|\psi_t\|^2 \\ &\quad = \left(f_1(u + \vartheta_1(\tilde{u} - u), \tilde{u}_t)\psi + f_2(u, u_t + \vartheta_2(\tilde{u}_t - u_t))\psi_t, \psi_t\right) \\ &\quad \leq (\text{by } (3) \text{ and Poincare inequality}) \\ &\quad \leq k_2\left(\sqrt{\lambda_1}\right)^{-1}\|\psi\| \cdot |\psi_t| + k_3|\psi_t|^2 \\ &\quad \leq c_1\left(|\psi_t|^2 + \|\psi\|^2\right), \end{aligned}$$

i.e.,

$$\frac{d}{dt}\left(|\psi_t|^2 + \|\psi\|^2\right) \leq 2c_1\left(|\psi_t|^2 + \|\psi\|^2\right),$$

where $c_1 > 0$, $\vartheta_1, \vartheta_2 \in (0,1)$. So, we have the Lipschitz property

$$\begin{aligned}\|\tilde{\psi}(t) - \psi(t)\|^2_{H_0^1 \times L^2} &= |\tilde{u}_t(t) - u_t(t)|^2 + \|\tilde{u}(t) - u(t)\|^2 \\ &\leq \exp(2c_1 t)\left(|\eta|^2 + \|\xi\|^2\right), \qquad \forall t \geq 0. \quad (14)\end{aligned}$$

Consider the difference $\theta = \tilde{u} - u - U$, with U the solution of the linearized equation (12) of (1)–(2). Obviously,

$$\theta(0) = \theta_t(0) = 0, \tag{15}$$

and

$$\theta_{tt} - \alpha\Delta\theta - \Delta\theta = f(\tilde{u}, \tilde{u}_t) - f(u, u_t) - f_1(u, u_t)U - f_2(u, u_t)U_t = h. \tag{16}$$

By the mean value theorem, we have

$$\begin{aligned} h = {} & \left[f_1(u + \vartheta_3(\tilde{u} - u), \tilde{u}_t) - f_1(u, \tilde{u}_t) + f_1(u, \tilde{u}_t) - f_1(u, u_t)\right](\tilde{u} - u) \\ & + \left[f_2(u, u_t + \vartheta_4(\tilde{u}_t - u_t)) - f_2(u, u_t)\right](\tilde{u}_t - u_t) \\ & - f_1(u, u_t)\theta - f_2(u, u_t)\theta_t, \end{aligned} \tag{17}$$

where $\vartheta_i \in (0,1)$, $i = 3, 4$.

Taking the scalar product of each side of (16) with θ_t in $L^2(\Omega)$ and by (15), we find

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(|\theta_t|^2 + \|\theta\|^2\right) + \alpha\|\theta_t\|^2 \\ & \quad = (h, \theta_t) \\ & \quad \leq \text{by } (3), (4) \\ & \quad \leq |\theta_t|\Big(\vartheta_3 k_4 |\tilde{u} - u|^{1+\delta_1} + k_4|\tilde{u} - u|\,|\tilde{u}_t - u_t|^{\delta_2} \\ & \qquad\qquad + \vartheta_4 k_5 |\tilde{u}_t - u_t|^{1+\delta_3} + k_1|\theta| + k_2|\theta_t|\Big), \end{aligned}$$

i.e.,

$$\frac{d}{dt}\left(|\theta_t(t)|^2 + \|\theta(t)\|^2\right)$$
$$\leq c_2\left(|\theta_t(t)|^2 + \|\theta(t)\|^2\right) + c_3\Big(\|\tilde{u}(t) - u(t)\|^{2+2\delta_1}$$
$$+ \|\tilde{u}(t) - u(t)\|^2 |\tilde{u}_t(t) - u_t(t)|^{2\delta_2} + |\tilde{u}_t(t) - u_t(t)|^{2+2\delta_3}\Big),$$

where $c_2 > 0, c_3 > 0$. By the Gronwall inequality and (14), we obtain

$$|\theta_t(t)|^2 + \|\theta(t)\|^2$$
$$\leq \frac{c_3}{c_2}\exp(c_2 t)\int_0^t \Big(\|\tilde{u}(s) - u(s)\|^{2+2\delta_1}$$
$$+ \|\tilde{u}(s) - u(s)\|^2 |\tilde{u}_t(s) - u_t(s)|^{2\delta_2} + |\tilde{u}_t(s) - u_t(s)|^{2+2\delta_3}\Big)\, ds$$
$$\leq c_4\left[\left(|\eta|^2 + \|\xi\|^2\right)^{1+\delta_1} + \left(|\eta|^2 + \|\xi\|^2\right)^{1+\delta_2} + \left(|\eta|^2 + \|\xi\|^2\right)^{1+\delta_3}\right]$$
$$\times \exp(c_5 t), \qquad \forall t \geq 0,$$

where $c_4 > 0, c_5 > 0$, that is,

$$|\tilde{\varphi}(t) - \varphi(t) - U(t)|^2_{H_0^1 \times L^2}$$
$$\leq c_4\Bigg[\left(\left|(\xi,\eta)^T\right|^2_{H_0^1 \times L^2}\right)^{1+\delta_1} + \left(\left|(\xi,\eta)^T\right|^2_{H_0^1 \times L^2}\right)^{1+\delta_2}$$
$$+ \left(\left|(\xi,\eta)^T\right|^2_{H_0^1 \times L^2}\right)^{1+\delta_3}\Bigg]\exp(c_5 t), \qquad \forall t \geq 0,$$

Therefore,

$$\frac{|\tilde{\varphi}(t) - \varphi(t) - U(t)|^2_{H_0^1 \times L^2}}{\left|(\xi,\eta)^T\right|^2_{H_0^1 \times L^2}}$$
$$\leq c_4\left[\left|(\xi,\eta)^T\right|^{1+\delta_1}_{H_0^1 \times L^2} + \left|(\xi,\eta)^T\right|^{1+\delta_3}_{H_0^1 \times L^2} + \left|(\xi,\eta)^T\right|^{1+\delta_3}_{H_0^1 \times L^2}\right]\exp(c_5 t)$$
$$\to 0 \quad \text{as } (\xi,\eta)^T \to 0 \quad \text{in } D(C). \tag{18}$$

Since $E = H_0^1(\Omega) \times L^2(\Omega)$ is dense in $D(C)$, (18) is true for solutions $\tilde{\varphi}(t), \varphi(t), U(t) \in E$. The proof is completed.

3. GLOBAL ATTRACTOR

In this section, we prove the existence of the global attractor of semigroup $S(t)$, $t \geq 0$ defined by (9).

First, we define a new weight inner product and norm in $E = H_0^1(\Omega) \times L^2(\Omega)$ as

$$(\varphi, \psi)_E = k((u_1, u_2)) + (v_1, v_2), \qquad |\varphi|_E = (\varphi, \varphi)_E^{1/2} \tag{19}$$

for $\varphi = (u_1, v_1)^T$, $\psi = (u_2, v_2)^T \in E$, where

$$k = \frac{4 + \alpha^2\lambda_1}{4 + 2\alpha^2\lambda_1} > 0. \tag{20}$$

Obviously, the norm $|\cdot|_E$ is equivalent to the usual norm $|\cdot|_{H_0^1 \times L^2}$ in E.

Let $\varphi = (u, v)^T$, $v = u_t + \varepsilon u$, where ε is chosen as

$$\varepsilon = \frac{\alpha\lambda_1}{4 + 2\alpha^2\lambda_1}, \tag{21}$$

then the system (1)–(2) or (9) can be written as

$$\varphi_t + \Lambda\varphi = F(\varphi), \qquad \varphi(0) = (u_0, u_1 + \varepsilon u_0)^T, \tag{22}$$

where

$$\begin{gathered} F(\varphi) = \begin{pmatrix} 0 \\ f(u, v - \varepsilon u) + g \end{pmatrix}, \\ \Lambda = \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon(\alpha A - \varepsilon)I & (\alpha A - \varepsilon)I \end{pmatrix}, \\ D(\Lambda) = D(A) \times D(A). \end{gathered} \tag{23}$$

LEMMA 3. *For any* $\varphi = (u, v)^T \in E$,

$$(\Lambda\varphi, \varphi)_E \geq \sigma|\varphi|_E^2 + \frac{\alpha\lambda_1}{2}|v|^2, \tag{24}$$

where

$$\sigma = \frac{\lambda_1\alpha}{4 + \alpha^2\lambda_1 + \alpha\sqrt{\alpha^2\lambda_1^2 + 4\lambda_1}}. \tag{25}$$

Proof. For any $\varphi = (u, v)^T \in D(\Lambda)$, from (19) and (23), we have

$$
\begin{aligned}
(\Lambda\varphi, \varphi)_E - \sigma|\psi|_E^2 - \frac{\alpha\lambda_1}{2}|v|^2 \\
&= \varepsilon k\|u\|^2 + \alpha(Av, v) - \varepsilon|v|^2 + \varepsilon^2(u, v) \\
&\quad - \sigma k\|u\|^2 - \sigma|v|^2 - \frac{\alpha\lambda_1}{2}|v|^2 \\
&\geq (\varepsilon - \sigma)k\|u\|^2 + \left(\frac{\alpha\lambda_1}{2} - \varepsilon - \sigma\right)|v|^2 - \frac{\varepsilon^2}{\sqrt{k\lambda_1}} \cdot \sqrt{k}\|u\| \cdot |v|.
\end{aligned} \tag{26}
$$

From (20), (21), and (25), elementary computations show

$$4(\varepsilon - \sigma)\left(\frac{\alpha\lambda_1}{2} - \varepsilon - \sigma\right) \geq \frac{\varepsilon^4}{k\lambda_1},$$

thus by (26),

$$(\Lambda\varphi, \varphi)_E \geq \sigma|\varphi|_E^2 + \frac{\alpha\lambda_1}{2}|v|^2 \quad \text{for} \quad \varphi = (u, v)^T \in D(\Lambda).$$

Since E is dense in $D(\Lambda)$, the proof is completed.

Now, we consider the absorbing property of the semigroup $S_\varepsilon(t)$, $t \geq 0$ defined by (22) on E.

Taking the scalar product of each side of (22) with $\varphi = (u, v)^T$ in E, we find

$$\frac{1}{2}\frac{d}{dt}|\varphi|_E^2 = -(\Lambda\varphi, \varphi)_E + (F(\varphi), \varphi). \tag{27}$$

By (24),

$$-2(\Lambda\varphi, \varphi)_E \leq -2\sigma|\varphi|_E^2 - \alpha\lambda_1|v|^2. \tag{28}$$

By (19) and (23),

$$
\begin{aligned}
2(F(\varphi), \varphi)_E &= 2(f(u, v - \varepsilon u) + g, v) \\
&\leq \text{by (3) and (Young inequality)} \\
&\leq \frac{(|g| + k_0)^2}{\alpha\lambda_1} + \alpha\lambda_1|v|^2 + \sigma|v|^2 + \sigma(k\|u\|^2) \\
&\quad + (1 - \delta_0)\left(\frac{(\delta_0\lambda_1)^{\delta_0} k_1^2}{k^{\delta_0}\sigma^{1+\delta_0}}\right)^{1/(1-\delta_0)}
\end{aligned}
$$

$$\leq \alpha\lambda_1|v|^2 + \sigma|\varphi|_E^2 + \frac{\left(|g| + k_0\right)^2}{\alpha\lambda_1}$$

$$+(1-\delta_0)\left(\frac{(\delta_0\lambda_1)^{\delta_0}k_1^2}{k^{\delta_0}\sigma^{1+\delta_0}}\right)^{1/(1-\delta_0)}. \tag{29}$$

Thus, by (27), (28), and (29),

$$\frac{d}{dt}|\varphi|_E^2 \leq -\sigma|\varphi|_E^2 + M_0,$$

where

$$M_0 = \frac{\left(|g| + k_0\right)^2}{\alpha\lambda_1} + (1-\delta_0)\left(\frac{(\delta_0\lambda_1)^{\delta_0}k_1^2}{k^{\delta_0}\sigma^{1+\delta_0}}\right)^{1/(1-\delta_0)}.$$

Applying Gronwall's inequality, we obtain the following absorbing inequality in the space $(E, |\cdot|_E)$:

$$|\varphi(t)|_E^2 \leq \left(\|u_0\|^2 + |u_1 + \varepsilon u_0|^2\right)\cdot\exp(-\sigma t) + \frac{M_0}{\sigma}[1 - \exp(-\sigma t)],$$

or

$$\limsup_{t\to+\infty}|\varphi(t)|_E^2 \leq \frac{M_0}{\sigma}, \tag{30}$$

As a direct consequence of the absorbing property (30), we have

LEMMA 4. *The map* $S_\varepsilon(t)$: $E \to E$, $(u_0, u_1 + \varepsilon u_0)^T \to (u(t), u_t(t) + \varepsilon u(t))^T$ *defined by* (22) *is point dissipative and bounded for any* $t > 0$.

LEMMA 5. *The map* $S(t)$: $E \to E$, $(u_0, u_1)^T \to (u(t), u_t(t))^T$ *defined by* (9) *is point dissipative and bounded for any* $t > 0$.

Proof. Since the norm $|\cdot|_E$ is equivalent to the usual norm $|\cdot|_{H_0^1\times L^2}$ in E and the relation of $S_\varepsilon(t)$, defined by (22), with $S(t)$ is made by the reversible transformation $u = u$, $v = u_t + \varepsilon u$, i.e.,

$$S_\varepsilon(t) = R_\varepsilon S(t) R_{-\varepsilon}, \tag{31}$$

where R_ε is an isomorphism of E

$$R_\varepsilon\colon \{u, v\} \to \{u, v + \varepsilon u\}.$$

By Lemma 4, the proof is completed.

THEOREM 2. *The nonlinear semigroup* $S(t)$, $t \geq 0$ *of* (9) *possesses a global attractor* $\mathfrak{B}$ *in* E.

Proof. By Lemma 5, (11) and e^{-Ct} is compact, we complete the proof of Theorem 2.

LEMMA 6. *For any orthonormal family of elements of* E, $\{(\xi_j, \eta_j)^T\}_{j=1}^{\ell}$, *we have*

$$\sum_{j=1}^{\ell} |\xi_j|^2 \leq k^{-1} \sum_{j=1}^{\ell} \lambda_j^{-1}. \tag{32}$$

Proof. Similar to the proof of Lemma VI. 6.3 in [9].

4. PROOF OF THEOREM 1

In this section, we estimate the Hausdorff dimension of the global attractor $\mathfrak{B}$ for (9) in E.

We consider the equivalent system (22). By (31) and $\mathfrak{B}$ is the global attractor of the semigroup $\{S(t), t \geq 0\}$ defined by (9), $R_\varepsilon \mathfrak{B}$ is the global attractor of $\{S_\varepsilon(t), t \geq 0\}$ defined by (22) and $R_\varepsilon \mathfrak{B}$, $\mathfrak{B}$ have the same dimension. So we need estimate the dimension of $R_\varepsilon \mathfrak{B}$ only. We consider the first variation equation of (22)

$$\Psi' = \left[-\Lambda + F'(\varphi)\right]\Psi \tag{33}$$

with the initial condition

$$\Psi(0) = (\xi, \eta)^T \in E, \tag{34}$$

where $\Psi = (U, V)^T$, $\varphi = (u, v)^T$ is a solution of (22),

$$F'(\varphi) = \begin{pmatrix} 0 & 0 \\ f_1'(u, v - \varepsilon u) - \varepsilon f_2'(u, v - \varepsilon u) & f_2'(u, v - \varepsilon u) \end{pmatrix} \tag{35}$$

Since the norm $|\cdot|_E$ is equivalent to the usual norm $|\cdot|_{H_0^1 \times L^2}$ in E and (31), it is easy to show from Lemma 2 that (33)–(34) is a well-posed problem in E, the mapping $S_\varepsilon(t)$ defined by (22) is Fréchet differentiable on E for any $t > 0$, its differential at $\varphi = (u_0, u_1 + \varepsilon u_0)^T$ is the linear operator on $E, (\xi, \eta)^T \mapsto (U(t), V(t))^T$, where $(U, V)^T$ is the solution of (33)–(34).

LEMMA 7. *Consider the system* (22) *with the conditions* (3) *and* (4). *Let* Φ *denote a set of* ℓ *vectors* $\{\Phi_1, \Phi_2, \dots, \Phi_\ell\}$ *which are orthonormal in* E. *If*

$$\sup_{\Phi \subset E} \sup_{\varphi \in R_\varepsilon \mathfrak{B}} \sum_{j=1}^{\ell} \Big((-\Lambda + F'(\varphi))\Phi_j, \Phi_j \Big)_E \leq 0, \tag{36}$$

then the Hausdorff dimension of the global attractor $\mathfrak{B}$ *is less than or equal to* ℓ.

Proof. This is a direct consequence of Theorem V. 3.3, Eqs. (V. 3.47)–(V. 3.49) and identity (VI. 6.24) of [9].

LEMMA 8. *If the function* $f(u, v)$ *satisfies conditions* (3), (4), *and* $\alpha > 2k_3/\lambda_1$, *then the Hausdorff dimension* $d_H(R_\varepsilon \mathfrak{B})$ *of the global attractor* $R_\varepsilon \mathfrak{B}$ *for system* (22) *in* E *satisfies*

$$d_H(R_\varepsilon \mathfrak{B}) \leq \min\left\{ \ell \,|\, \ell \in N, \ \frac{1}{\ell} \sum_{j=1}^{\ell} \lambda_j^{-1} \leq \frac{2\delta\alpha\sigma}{k_2 + \varepsilon k_3} \right\}, \tag{37}$$

where $\varepsilon, \delta, \sigma$ *are defined by* (6).

Proof. Let $\ell \in N$ be fixed. Consider ℓ solutions $\Psi_1, \Psi_2, \dots, \Psi_\ell$ of (33)–(34). At a given time τ, let $Q_\ell(\tau)$ denote the orthogonal projector in E onto the space spanned by $\Psi_1, \Psi_2, \dots, \Psi_\ell$. Let $\Phi_j(\tau) = (\xi_j, \eta_j)^T \in E$, $j = 1, 2, \dots, \ell$, be an orthonormal basis of $Q_\ell(\tau)E$. Consider the quality

$$\sum_{j=1}^{\ell} \big((-\Lambda + F'(\varphi))\Phi_j, \Phi_j\big)_E = -\sum_{j=1}^{\ell} \big(\Lambda\Phi_j, \Phi_j\big)_E + \sum_{j=1}^{\ell} \big(F'(\varphi)\Phi_j, \Phi_j\big)_E.$$

By Lemma 3 and $|\Phi_j|_E = 1$,

$$-\big(\Lambda\Phi_j, \Phi_j\big)_E \leq -\sigma - \frac{\alpha\lambda_1}{2}|\eta_j|^2.$$

From (19) and (35),

$$\begin{aligned}
\left|\big(F'(\varphi)\Phi_j, \Phi_j\big)_E\right| &= \big((f_1'(u, u_t) - \varepsilon f_2'(u, u_t))\xi_j + f_2'(u, u_t)\eta_j, \eta_j\big) \\
&\leq \text{by } (3) \\
&\leq (k_2 + \varepsilon k_3)|\xi_j| \cdot |\eta_j| + k_3|\eta_j|^2 \\
&\leq \frac{(k_2 + \varepsilon k_3)}{4\delta\alpha}|\xi_j|^2 + \big[\delta\alpha(k_2 + \varepsilon k_3) + k_3\big]|\eta_j|^2,
\end{aligned}$$

hence,

$$\sum_{j=1}^{\ell}\big((-\Lambda+F'(\varphi))\Phi_j,\Phi_j\big)_E$$

$$\leq \sum_{j=1}^{\ell}\Big[-\sigma-\frac{\alpha\lambda_1}{2}|\eta_j|^2+\frac{k_2+\varepsilon k_3}{4\delta\alpha}|\xi_j|^2$$

$$+\big(\delta\alpha(k_2+\varepsilon k_3)+k_3\big)|\eta_j|^2\Big]$$

$$\leq \sum_{j=1}^{\ell}\left[-\sigma+\frac{k_2+\varepsilon k_3}{4\delta\alpha}|\xi_j|^2-\left(\frac{\alpha\lambda_1}{2}-\delta\alpha(k_2+\varepsilon k_3)-k_3\right)|\eta_j|^2\right]$$

$\leq$ by (32)

$$\leq \sum_{j=1}^{\ell}\Big[-\sigma+\frac{k_2+\varepsilon k_3}{4\delta\alpha}\frac{1}{k}\lambda_j^{-1}-\Big(\frac{\alpha\lambda_1}{2}$$

$$-\delta\alpha(k_2+\varepsilon k_3)-k_3\Big)|\eta_j|^2\Big].$$

If

$$\alpha>\frac{2k_3}{\lambda_1},\qquad \frac{1}{\ell}\sum_{j=1}^{\ell}\lambda_j^{-1}\leq k\frac{4\delta\alpha\sigma}{k_2+\varepsilon k_3},\tag{38}$$

then,

$$\sum_{j=1}^{\ell}\big((-\Lambda+F'(\varphi))\Phi_j,\Phi_j\big)_E\leq 0.$$

By Lemma 7, the Hausdorff dimension $d_H(R_\varepsilon\mathfrak{B})\leq\ell$.

Since

$$\tfrac{1}{2}<k<1,\tag{39}$$

by (38), the Lemma 6 is proved.

COROLLARY 1. *If the function f is independent of u_t and satisfies conditions* (7), *then the Hausdorff dimension $d_H(R_\varepsilon\mathfrak{B})$ of the global attractor $R_\varepsilon\mathfrak{B}$*

for system (22) *in* E *satisfies*

$$d_H(R_\varepsilon \mathfrak{B}) \le \min\left\{ \ell \,|\, \ell \in N,\ \frac{1}{\ell}\sum_{j=1}^{\ell} \lambda_j^{-1} \le \frac{\alpha\lambda_1\sigma}{k_2^2} \right\}. \tag{40}$$

Proof. It is a special case of Lemma 8 in which $k_3 = 0$.

COROLLARY 2. *If the function* $f(u,v)$ *satisfies* (3), (4), *and* $-k_3 \le f_2(u,v) \le 0$, $\forall (u,v) \in R \times R$, *then the Hausdorff dimension* $d_H(R_\varepsilon \mathfrak{B})$ *of the global attractor* $R_\varepsilon \mathfrak{B}$ *for system* (22) *in* E *satisfies*

$$d_H(R_\varepsilon \mathfrak{B}) \le \min\left\{ \ell \,|\, \ell \in N,\ \frac{1}{\ell}\sum_{j=1}^{\ell} \lambda_j^{-1} \le \frac{\alpha\lambda_1\sigma}{(k_2 + \varepsilon k_3)^2} \right\}. \tag{41}$$

Combining with Lemma 8, Corollary 1, and (6), the proof is completed.

REFERENCES

1. A. C. Scott, F. Y. F. Chu, and S. A. Reible, Magnetic-flux propagation on a Josephson transmission line, *J. Appl. Phys.* **47** (1976), 3272–3286.
2. P. S. Lomdahl, O. H. Soerensen, and P. L. Christiansen, Soliton excitation in Josephson tunnel junctions, *Phys. Rev. B* (3), **25** (1982), 5737–5748.
3. G. F. Webb, Existence and asymptotic behavior for a strongly damped nonlinear wave equation, *Canad. J. Math.* **32** (1980), 631–643.
4. P. Massatt, Limiting behavior for strongly damped nonlinear wave equations, *J. Differential Equations* **48** (1983), 334–349.
5. J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Amer. Math. Soc., Providence, Rhode Island, 1988.
6. S. Zhu and S. Zhou, Dimension of the global attractor for damped sine-Gordon equation, Nonlinear Anal. To appear.
7. G. Wang and S. Zhu, On dimension of the global attractor for damped sine-Gordon equation, *J. Math. Phys.* **38** (1997), 6:3137–3141.
8. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," *Appl. Math. Sci.*, Vol. 44, Springer-Verlag, New York, 1983.
9. R. Témam, "Infinite-dimensional Dynamical Systems in Mechanics and Physics," *Appl. Math. Sci.* **68**, Springer-Verlag, New York, 1988.